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Theory of Liquid Ink Development in Electrophotography

Abstract: When an electric field is present across two different fluid dielectrics having a common plane boundary, some disturbances in the interfacial boundary are found to grow in time. The liquid ink development process is viewed as the result of varying instability of the oil-ink interface as a function of the differing field gradient in light and dark areas of the exposed image. In an analysis of the disturbances into normal modes, the theory relates the effects upon instability due to potential difference across the oil, the surface tension, the respective viscosities of the oil and the ink, and the finite thicknesses of the oil, ink, and the photoconductor layer. The threshold potential difference for the onset of instability is also given.

Introduction

The formation of a latent electrostatic image in electrophotography is well understood [1, 2]. In the liquid ink development process, which is described in [3], a thin layer of liquid ink is pressure deposited over a thin layer of dielectric oil already resident on the photoconductor (PC), on which a latent electrostatic image has been formed. After development, the ink breaks through the oil layer in the image area but not in the exposed area. The purpose of this paper is to analyze this development process and to relate the physical and material parameters that effect the process.

The liquid ink development process is viewed as the result of instability in the oil-ink interface; i.e., when an electric field is present across the layers of oil and ink, some disturbances in the interface may grow in time. After a sufficient time interval in the developing stage, the ink may eventually break through the thin oil film and displace it next to the PC.

In the mathematical formulation discussed here, the disturbances are analyzed into normal modes. This concept is widely adopted in most stability analyses [4]. Melcher [7] and Michael [8] applied the analysis to plane surface gravity waves with a normal electrostatic field at the free surface of a conducting fluid. In the liquid ink development process, however, we are interested in the plane surface waves at the interface between the conducting ink and the dielectric oil. In this case, the surface waves are free from gravitational field (neglecting centrifugal force). For infinitesimal disturbances, the problem consists of linearized hydrodynamic equations of continuity and of motion, and boundary conditions. In addition, the electrostatic problem is also solved in its linearized form. The solution to the coupled hydrodynamic and electrostatic problems results in a characteristic equation, from which the growth rates for all modes of disturbances may be computed.

A qualitative understanding of the interfacial instability can first be derived from the simplest case for inviscid oil and ink. The effect upon instability due to the potential difference across the oil can then be studied. The disturbances are found to be unstable only for a finite range of wave numbers. There exists a wave number at which the growth rate is maximum. For each mode, there also exists a threshold potential difference across the oil, below which all disturbances are stable, i.e., not growing. These results of threshold potential differences were later found valid also for the cases when oil and ink viscous effects are included.

The viscosity of the oil is several orders of magnitude less than that of the ink. Extensive computations are carried out for the case of viscous ink, neglecting the viscosity of the oil. The results are plotted showing the stabilizing effect. The maximum growth rate and its corresponding wave number are also plotted. All results are presented in terms of dimensionless quantities. When the viscosities of both the ink and the oil are taken into consideration, the following contrasting phenomenon is observed: The viscosity of the ink has little effect on the disturbance growth at small wave numbers, but it drastically reduces the growth rate of disturbances at large wave numbers. On the contrary, the viscosity of the oil stabilizes the disturbances with small wave numbers and has little effect on those with large wave numbers.

Mathematical formulation

In the liquid ink development process, a thin layer of ink is pressed upon a thin film of oil previously applied onto the PC. The configuration is shown schematically in

Fig. 1. The mathematical analysis will be based on the following assumptions:

1. The ink is a perfect conductor and is held at constant voltage V_i.
2. The substrate is a perfect conductor, and is grounded; hence $\Psi \equiv 0$, where Ψ is the electric potential.
3. The latent electrostatic image is characterized by the constant charge q_0 uniformly distributed at the oil-PC interface.
4. The PC is elastically rigid.
5. The pressure p_0, imposed by the soft elastomer, is constant and uniformly distributed at the ink-elastomer interface.
6. The oil and ink are homogeneous, incompressible, and viscous.
7. The dielectric properties of oil and PC are homogeneous.

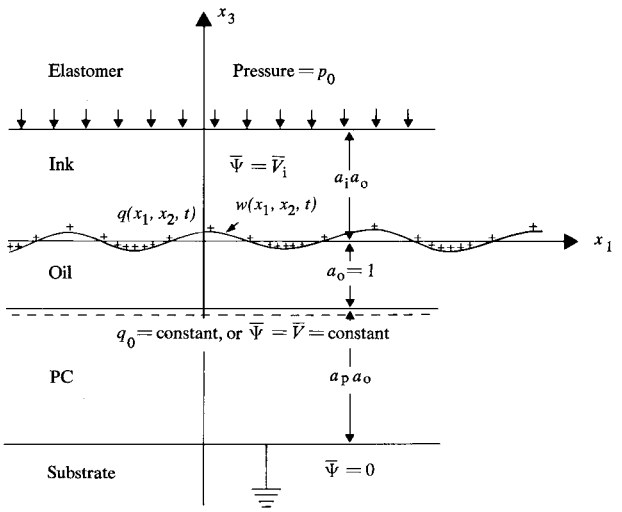

Figure 1 Liquid ink development for a multilayer configuration on a photoconductor film.

• *Notation*

Let the $x_1 - x_2$ plane be coincident with the undisturbed oil-ink interface as shown in Fig. 1. Normalize all linear dimensions with respect to the nominal thickness of the oil film, a_0. The following notation is used in the analysis:

a_o	thickness of the oil;
a_i	thickness ratio, ink vs oil;
a_p	thickness ratio, PC vs oil;
$\rho, \bar{\rho}$	densities of the oil and the ink, respectively;
$\mu, \bar{\mu}$	viscosities of the oil and the ink, respectively;
$u_i, \bar{u}_i$	velocity components of the oil and the ink, respectively;
$\sigma_{ij}, \bar{\sigma}_{ij}$	stress tensors of the oil and the ink, respectively;
$p, \bar{p}$	pressures of the oil and the ink, respectively;
$w(x_1, x_2, t)$	vertical dimensional change of oil-ink interface from the undisturbed state;
η_i	unit normal vector of the oil-ink interface;
$\varepsilon_{oil}, \varepsilon_{PC}$	dielectric constants of the oil and the PC, respectively;
$\Psi(x_1, x_2, x_3, t)$	electric potential;
q_0	constant charge uniformly distributed at the oil-PC interface;
V	constant voltage (corresponding to q_0) at the oil-PC interface;
$q(x_1, x_2, x_3, t)$	charge distributed at the oil-ink interface;
V_i	constant voltage in the ink;
α	oil-ink surface tension.

In addition, the summation convention for repeated indices is understood throughout the analysis. The comma (,) indicates differentiation with respect to the respective coordinate. The Latin subscripts $i, j, k, \cdots$, take the indices 1, 2, and 3, and the Greek subscripts $\alpha, \beta, \cdots$, take the indices 1 and 2 in the $x_1 - x_2$ plane. An overdot denotes material differentiation, which, for the linearized analysis carried out here, can be identified with time differentiation.

• *The electrostatic problem*

The electrostatic potential $\Psi(x_1, x_2, x_3, t)$ satisfies the Laplace equation

$$\Psi_{,ii} = 0. \tag{1}$$

The boundary conditions specify that 1) the substrate is grounded; 2) the ink, a perfect conductor, is held at constant potential V_i; 3) Ψ must be continuous at the oil-PC interface; and 4) the charge at the oil-PC interface is a constant, q_0. The mathematical expressions for these conditions are

$$\Psi = 0, \qquad x_3 = -(1 + a_p), \tag{2a}$$

$$\Psi = V_i, \qquad x_3 = w/a_o, \tag{2b}$$

$$\Psi(x_1, x_2, x_3 = -1 - 0, t) = \Psi(x_1, x_2, x_3 = -1 + 0, t), \tag{2c}$$

$$-\varepsilon_{oil}\Psi_{,3}(x_1, x_2, x_3 = -1 + 0, t) + \varepsilon_{PC}\Psi_{,3}(x_1, x_2, x_3 = -1 - 0, t) = 4\pi a_o q_0. \tag{2d}$$

The charge distribution at the oil-ink interface, q, can be computed from Ψ, according to

$$4\pi a_o q = \varepsilon_{oil}\Psi_{,3}(x_1, x_2, x_3 = w/q_0, t). \tag{3}$$

• *The coupled electrostatic-hydrodynamic problem*

For incompressible oil and ink, the continuity equations are, respectively,

$$u_{i,i} = \bar{u}_{i,i} = 0. \tag{4}$$

The equations of motion are, respectively,

$$\rho a_0 \ddot{u}_i = \sigma_{ij,j}, \tag{5a}$$

$$\bar{\rho} a_0 \ddot{\bar{u}}_i = \bar{\sigma}_{ij,j}, \tag{5b}$$

where the stress tensors σ_{ij} and $\bar{\sigma}_{ij}$ are, respectively [5],

$$\sigma_{ij} = -p\delta_{ij} + (\mu/a_0)(u_{i,j} + u_{j,i}) + (\varepsilon_{oil}/4\pi a_0^2)(\Psi_{,i}\Psi_{,j} - \tfrac{1}{2}\Psi_{,k}\Psi_{,k}\delta_{ij}), \tag{6a}$$

$$\bar{\sigma}_{ij} = -\bar{p}\delta_{ij} + (\bar{\mu}/a_0)(\bar{u}_{i,j} + \bar{u}_{j,i}), \tag{6b}$$

where δ_{ij} is the Kronecker delta.

The solution to the Eqs. (4) and (5) must satisfy the following boundary conditions:

1. Assuming no shear stress at the ink-elastomer interface, $x_3 = a_i$,

$$\bar{\sigma}_{i3} = -p_0\delta_{i3}, \tag{7}$$

2. at the oil-PC interface, $x_3 = -1$, for a rigid PC,

$$u_i = 0, \tag{8}$$

3. at the oil-ink interface, $x_3 = w/a_0$, the velocities must be continuous, i.e.,

$$u_i = \bar{u}_i, \tag{9}$$

and the equilibrium of surface tractions has the form

$$(\sigma_{ij} - \bar{\sigma}_{ij})\eta_j = -(q/a_0)\Psi_{,i} + (\alpha/a_0^2)\eta_i w_{,\beta\beta}. \tag{10}$$

In Eq. (10), the first term on the right-hand side represents the surface traction due to the charge distribution, q, at the oil-ink interface [5], and the second term on the right-hand side is the Laplace formula [6]. It is also implied in Eq. (10) that $|w| \ll a_0$, and that both components $|w_{,\beta}|$ are small compared with unity. Therefore, this linear analysis describes only the initial growth of disturbances. When growth is sufficient to break through the oil, the effect is likely to be nonlinear. Consequently, the unit normal vector, η_i, can be evaluated, approximately, according to

$$\eta_i \approx (-w_{,1}/a_0, -w_{,2}/a_0, 1). \tag{11}$$

The oil-ink interface $x_3 = w(x_1, x_2, t)$ is related to u_3 (or $\bar{u}_3$) according to

$$\dot{w}(x_1, x_2, t) = u_3(x_1, x_2, x_3 = 0, t) = \bar{u}_3(x_1, x_2, x_3 = 0, t). \tag{12}$$

Normal mode analysis

• *The equilibrium solution*

One solution to the problem posed by Eqs. (1) through (12) can be written down immediately as

$$u_i(x_1, x_2, x_3, t) \equiv \bar{u}_i(x_1, x_2, x_3, t) \equiv w(x_1, x_2, t) \equiv 0, \tag{13}$$

$$\Psi(x_1, x_2, x_3, t) = V_i + (V_i - V)x_3, -1 \le x_3 \le w/a_0,$$
$$= (V/a_p)[(1 + a_p) + x_3], -(1 + a_p) \le x_3 \le -1, \tag{14}$$

where V and q_0 are related through the Eq. (2d), i.e.,

$$-\varepsilon_{oil}(V_i - V) + \varepsilon_{PC}(V/a_p) = 4\pi a_0 q_0. \tag{15}$$

The charge distribution at the oil-ink interface, q, is given by

$$4\pi a_0 q = \varepsilon_{oil}(V_i - V), \tag{16}$$

which is a constant. The pressure fields are

$$p = p_0 + \tfrac{3}{2}(\varepsilon_{oil}/4\pi a_0^2)(V_i - V)^2,$$
$$\bar{p} = p_0. \tag{17}$$

This solution is trivially true. It represents an equilibrium state of the system. We now ask: What would happen as a result of any infinitesimal disturbance superposed onto this equilibrium state?

• *Linearization in the normal mode analysis and the electrostatic solution*

Analyzing the disturbances into normal modes, we seek solutions whose dependence on x_1, x_2, and t is given by

$$E \equiv E(x_1, x_2, t) \equiv \exp[ik_\alpha x_\alpha + nt], \tag{18}$$

where k_α, $\alpha = 1, 2$, are constants representing wave numbers in the x_α direction, and n is the growth rate. Thus, we write solutions in the form:

$$u_i(x_1, x_2, x_3, t) = \delta E v_i(x_3), \tag{19a}$$

$$\bar{u}_i(x_1, x_2, x_3, t) = \delta E \bar{v}_i(x_3), \tag{19b}$$

$$p(x_1, x_2, x_3, t) = p_0 + \tfrac{3}{2}(\varepsilon_{oil}/4\pi a_0^2)(V_i - V)^2 + \delta E\phi(x_3), \tag{19c}$$

$$\bar{p}(x_1, x_2, x_3, t) = p_0 + \delta E\bar{\phi}(x_3), \tag{19d}$$

$$\Psi(x_1, x_2, x_3, t) = V_i + (V_i - V)x_3 - \delta E(V_i - V)\psi_1(x_3), \quad -1 \le x_3 \le w/a_0,$$
$$= (V/a_p)[(1 + a_p) + x_3] - \delta E(V_i - V)\psi_2(x_3), \quad -(1 + a_p) \le x_3 \le -1, \tag{19e}$$

where $v_i(x_3)$, $\bar{v}_i(x_3)$, $\phi(x_3)$, $\bar{\phi}(x_3)$ and $\psi_1(x_3)$, $\psi_2(x_3)$ are unknown functions of x_3 as yet to be determined, and δ is a small numerical perturbation parameter, $|\delta| \ll 1$. The oil-ink interface, $x_3 = w(x_1, x_2, t)$ is, according to the Eq. (12), given by

$$w(x_1, x_2, t) = \int^t u_3(x_1, x_2, x_3 = 0, t')\, dt' = \delta E v_3(x_3 = 0)/n, \tag{20}$$

which is of the order δ.

Substituting the Eqs. (19e) and (20) into the Eqs. (1) and (2) of the electrostatic problem, we find that the functions $\psi_1(x_3)$ and $\psi_2(x_3)$ must satisfy the ordinary differential equations:

$$\psi_1''(x_3) - k^2\psi_1(x_3) = 0, \qquad -1 \le x_3 \le w/a_0, \tag{21a}$$

$$\psi_2''(x_3) - k^2\psi_2(x_3) = 0, \qquad -(1 + a_p) \le x_3 \le -1, \tag{21b}$$

where

$$k^2 \equiv k_\alpha k_\alpha (\equiv k_1^2 + k_2^2). \tag{22}$$

The linear boundary conditions (2a, c, and d) become

$$\psi_2(-1 - a_p) = 0, \tag{23a}$$

$$\psi_1(-1) = \psi_2(-1), \tag{23b}$$

$$\varepsilon_{oil}\psi_1'(-1) = \varepsilon_{PC}\psi_2'(-1). \tag{23c}$$

The boundary condition (2b) is, however, nonlinear when applied at the moving boundary $x_3 = w(x_1, x_2, t)/a_0$. For small δ, it is, therefore, linearized in the following manner. Using the Eq. (20) and the Taylor's expansion for small δ, we write

$$\psi_1(w/a_0) \approx \psi_1(0) + \delta E v_3(0)\psi_1'(0)/na_0 + 0(\delta^2),$$

where terms of orders higher than δ are omitted. Consequently, to the first order of δ, Eq. (19e) gives

$$\Psi(x_1, x_2, x_3 = w/a_0, t) \approx V_i + \delta E(V_i - V)\{[v_3(0)/na_0] - \psi_1(0)\}.$$

The boundary condition (2b) then leads to the condition

$$\psi_1(0) = v_3(0)/na_0. \tag{23d}$$

It should be remarked here that the above linearization procedure is applied throughout the analysis whenever a nonlinear boundary condition is encountered at the moving boundary $x_3 = w(x_1, x_2, t)/a_0$.

The electrostatic solution is uniquely determined by Eqs. (19e), (21) and (23) in terms of $v_3(0)$. The charge distribution, $q(x_1, x_2, t)$ at the oil-ink interface is found from Eq. (3) to be

$$4\pi a_0 q = \varepsilon_{oil}(V_i - V)\left\{1 - \delta E\,\frac{kv_3(0)}{na_0} \times \left[\frac{\varepsilon_{PC} + \varepsilon_{oil}(\tanh k)(\tanh ka_p)}{\varepsilon_{oil}\tanh ka_p + \varepsilon_{PC}\tanh k}\right]\right\}. \tag{24}$$

• *Linearized normal mode analysis*

In a similar manner, Eqs. (4), (5), and (6) lead to the coupled ordinary differential equations

$$v_3'(x_3) + ik_\alpha v_\alpha(x_3) = 0, \tag{25a}$$

$$\rho a_0 n v_\alpha(x_3) = -ik_\alpha\phi(x_3) + (\mu/a_0)[v_\alpha''(x_3) - k^2 v_\alpha(x_3)], \tag{25b}$$

$$\rho a_0 n v_3(x_3) = -\phi'(x_3) + (\mu/a_0)[v_3''(x_3) - k^2 v_3(x_3)], \tag{25c}$$

$$\bar{v}_3'(x_3) + ik_\alpha \bar{v}_\alpha(x_3) = 0, \tag{26a}$$

$$\bar{\rho} a_0 n \bar{v}_\alpha(x_3) = -ik_\alpha\bar{\phi}(x_3) + (\bar{\mu}/a_0)[\bar{v}_\alpha''(x_3) - k^2\bar{v}_\alpha(x_3)], \tag{26b}$$

$$\bar{\rho} a_0 n \bar{v}_3(x_3) = -\bar{\phi}'(x_3) + (\bar{\mu}/a_0)[\bar{v}_3''(x_3) - k^2\bar{v}_3(x_3)]. \tag{26c}$$

Making use of Eqs. (6), (20), and (24), the boundary conditions (7), (8), (9), and (10) can be expressed in terms of the functions $v_i(x_3)$, $\bar{v}_i(x_3)$ and $\phi(x_3)$, $\bar{\phi}(x_3)$. However, the algebraic manipulation can be somewhat simplified if we notice from the form (18) of the solution that it must be an invariant under an arbitrary rotation of the coordinate system about the x_3 axis. Accordingly, a contraction in the indices $\alpha = 1, 2$ in the $x_1 - x_2$ plane is performed. As a result, it can be shown, to the first order of δ, the boundary conditions become

$$\bar{v}_3''(a_i) + k^2\bar{v}_3(a_i) = 0, \tag{27a}$$

$$2(\bar{\mu}/a_0)\bar{v}_3'(a_i) = \bar{\phi}(a_i), \tag{27b}$$

$$v_3'(-1) = 0, \tag{28a}$$

$$v_3(-1) = 0, \tag{28b}$$

$$v_3'(0) = \bar{v}_3'(0), \tag{29a}$$

$$v_3(0) = \bar{v}_3(0), \tag{29b}$$

$$k^2[\mu v_3(0) - \bar{\mu}\bar{v}_3(0)] + [\mu v_3''(0) - \bar{\mu}\bar{v}_3''(0)] = 0, \tag{30a}$$

$$[\phi(0) - \bar{\phi}(0)] - 2[(\mu/a_0)v_3'(0) - (\bar{\mu}/a_0)\bar{v}_3'(0)] + (\alpha/na_0^2)f(k)kv_3(0) = 0, \tag{30b}$$

where $f(k)$ is the function of k defined by

$$f(k) \equiv \frac{3\varepsilon_{oil}(V_i - V)^2}{4\pi\alpha a_0} \times \left[\frac{\varepsilon_{PC} + \varepsilon_{oil}(\tanh k)(\tanh ka_p)}{\varepsilon_{PC}\tanh k + \varepsilon_{oil}\tanh ka_p}\right] - k. \tag{30c}$$

These boundary conditions do not involve $v_\alpha(x_3)$ and $\bar{v}_\alpha(x_3)$, $\alpha = 1, 2$. Hence, only the solutions $v_3(x_3)$, $\bar{v}_3(x_3)$ and $\phi(x_3)$ are needed. Their differential equations are obtained from the Eqs. (25) and (26), and have the form

$$\phi''(x_3) - k^2\phi(x_3) = 0, \tag{31a}$$

$$v_3''(x_3) - \lambda^2 v_3(x_3) = (a_0/\mu)\phi'(x_3), \tag{31b}$$

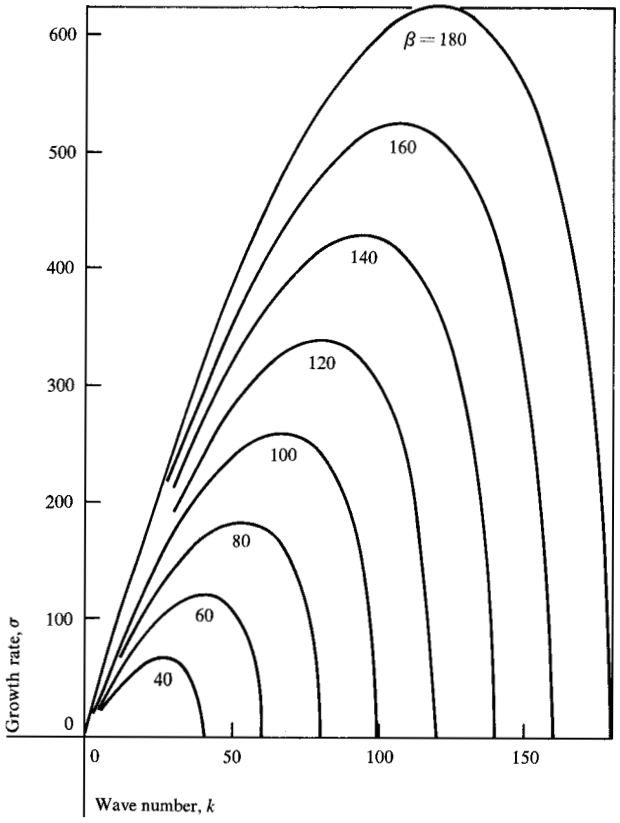

Figure 2 Dimensionless growth rate of unstable disturbances for the interface between inviscid oil and ink.

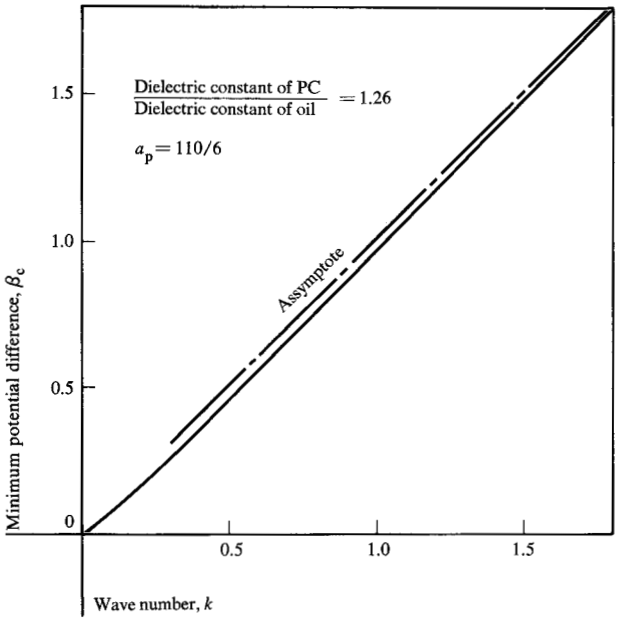

Figure 3 Minimum potential difference β_c for the existence of unstable disturbances.

$$\bar{\phi}''(x_3) - k^2\bar{\phi}(x_3) = 0, \tag{32a}$$

$$\bar{v}_3''(x_3) - \bar{\lambda}^2\bar{v}_3(x_3) = (a_0/\bar{\mu})\bar{\phi}'(x_3), \tag{32b}$$

where

$$\lambda^2 = k^2 + (\rho a_0^2 n/\mu), \tag{33a}$$

$$\bar{\lambda}^2 = k^2 + (\bar{\rho} a_0^2 n/\bar{\mu}). \tag{33b}$$

The problem now consists of the homogeneous differential equations (31) and (32) and the eight homogeneous boundary conditions (27) through (30). Consequently, the solution leads to the characteristic equation from which the growth rate, n, may be computed for a given wave number k. The characteristic equation is, in general, a transcendental relationship between n and k, and is in the form of an 8×8 determinant. Details of the analytic results will be omitted except for the case of inviscid oil and ink, which is described in the next section.

Results

All results will be presented in terms of the following dimensionless quantities:

$$k^2 = k_\alpha k_\alpha \equiv k_1^2 + k_2^2,$$

$$\beta = (3\varepsilon_{oil}/4\pi\alpha a_0)(V_i - V)^2,$$

$$\sigma^2 = \rho a_0^3 n^2/\alpha,$$

$$l_{ink} = (\bar{\rho}/\bar{\mu})(\alpha a_0/\rho)^{\frac{1}{2}},$$

$$l_{oil} = (\rho/\mu)(\alpha a_0/\rho)^{\frac{1}{2}}. \tag{34}$$

The definition of $f(k)$, Eq. (31C), becomes

$$f(k) \equiv \beta\left[\frac{\varepsilon_{PC} + \varepsilon_{oil}(\tanh k)(\tanh ka_p)}{\varepsilon_{PC}\tanh k + \varepsilon_{oil}\tanh ka_p}\right] - k, \tag{35}$$

and the equations (33) become

$$\lambda^2 = k^2 + \sigma l_{oil}, \tag{36}$$

$$\bar{\lambda}^2 = k^2 + \sigma l_{ink}.$$

Computations have been carried out using the following numerical data:

ρ	density of the oil = 0.92 gm/cc;
$\bar{\rho}$	density of the ink = 1.103 gm/cc;
a_0	thickness of the oil = 0.6×10^{-4} cm;
a_i	thickness ratio, ink/oil = 8/0.6;
a_p	thickness ratio, PC/oil = 11/0.6;
α	surface tension, oil-ink = 18 dynes/cm;
ε_{oil}	dielectric constant of oil = $3.4 \times \varepsilon_0$;
ε_{PC}	dielectric constant of PC = $2.7 \times \varepsilon_0$;
ε_0	dielectric constant in vacuum = 8.85×10^{-21} coulomb^2/dyne-cm^2;
μ/ρ	kinematic viscosity of oil $\approx$ 0.05 stokes;
$\bar{\mu}/\bar{\rho}$	kinematic viscosity of ink $\approx$ 330 stokes.

The real dimensional growth rate, n, may be obtained from σ according to

$$n = \sigma(\alpha/\rho a_o^3)^{\frac{1}{2}} = 0.95 \times 10^7 \sigma, \qquad 1/\text{s},$$

and the wavelength corresponding to the wave number k is

$$\text{wavelength} = 2\pi a_o/k = 3.8 \times 10^{-4}/k, \qquad \text{cm}.$$

The potential difference across the oil may be obtained from β by the following relationship:

$$(V_i - V) = [(4\pi\alpha a_o/3\varepsilon_{oil})\beta]^{\frac{1}{2}} = 43.5 \times \sqrt{\beta}, \text{ volts}.$$

The dimensionless parameters l_{oil} and l_{ink} are found to be

$$l_{ink} \approx 10^{-4},$$

$$l_{oil} \approx 0.7.$$

• *Inviscid oil and ink*

For the case of inviscid oil and ink, $\mu = \bar{\mu} = 0$, the solution may be derived directly from the Eqs. (31), (32), they become

$$\begin{aligned}
&\phi''(x_3) - k^2\phi(x_3) = 0,\\
&\rho a_o n v_3(x_3) = -\phi'(x_3),\\
&\bar{\phi}''(x_3) - k^2\bar{\phi}(x_3) = 0,\\
&\bar{\rho} a_o n \bar{v}_3(x_3) = -\bar{\phi}'(x_3),
\end{aligned} \tag{37}$$

and the boundary conditions that are needed in this case are Eqs. (27b), (28b), (29b) and (30b). They are

$$\begin{aligned}
&\bar{\phi}(a_i) = 0,\\
&v_3(-1) = 0,\\
&v_3(0) = \bar{v}_3(0),\\
&[\phi(0) - \bar{\phi}(0)] + (\alpha/na_o^2) f(k) k v_3(0) = 0.
\end{aligned} \tag{38}$$

The solution of the above homogeneous system leads to the following characteristic equation:

$$\sigma^2 = \frac{\rho k^2 \tanh k}{\rho + \bar{\rho}(\tanh k)(\tanh k a_i)} f(k), \tag{39}$$

where $f(k)$ is given in Eq. (35). The real valued function $f(k)$ is found to be positive, i.e., $f(k) > 0$, for a finite range of k, $0 < k < k_c$, where $f(k_c) = 0$; and $f(k) \leq 0$, when $k \geq k_c$. For large $\beta \gg 1$, the root k_c of $f(k_c) = 0$ is found to be $k_c \approx \beta$; while for small β, $0 < \beta \ll 1$, $k_c \approx \sqrt{\beta}$.

The simplicity of formula (39) permits the following conclusions:

1. σ is real for a finite range of the wave number k, $0 < k < k_c$. The positive root $\sigma > 0$ indicates that the wave grows in time, hence it is unstable. (The negative root $\sigma < 0$, which signifies the attenuation of the wave, is of no interest to us. Hence it is ignored.) The growth rate of the unstable waves are plotted in Fig. 2.
2. σ is purely imaginary when $k > k_c$. All waves whose wave number $k \geq k_c$ are stable. Hence, k_c is the cut-off wave number.
3. For a given wave number k, there exists a threshold potential difference, β_c, such that the wave is unstable only when the applied potential difference β exceeds β_c, $\beta > \beta_c$. The threshold β_c is found to be

$$\beta_c = k \frac{\varepsilon_{PC}\tanh k + \varepsilon_{oil}\tanh k a_p}{\varepsilon_{PC} + \varepsilon_{oil}(\tanh k)(\tanh k a_p)}, \tag{40}$$

   and is given in the plot of the Fig. 3.
4. For a given β, there exists a wave number k_{max} whose growth rate σ is a maximum called σ_{max}.

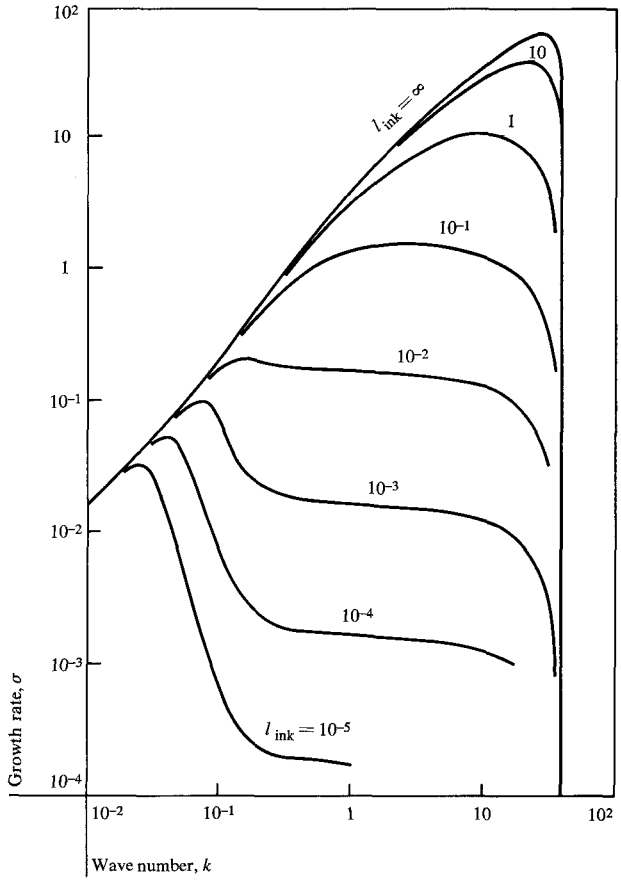

Figure 4 Dimensionless growth rate of unstable disturbances for $\beta = 40$ and inviscid oil.

• *Viscous ink and inviscid oil*

For the next order of complexity, we consider the viscous effect of the ink but neglect that of the oil. Thus, $\mu = 0$, $\bar{\mu} > 0$. The differential equations (31) and (32) are

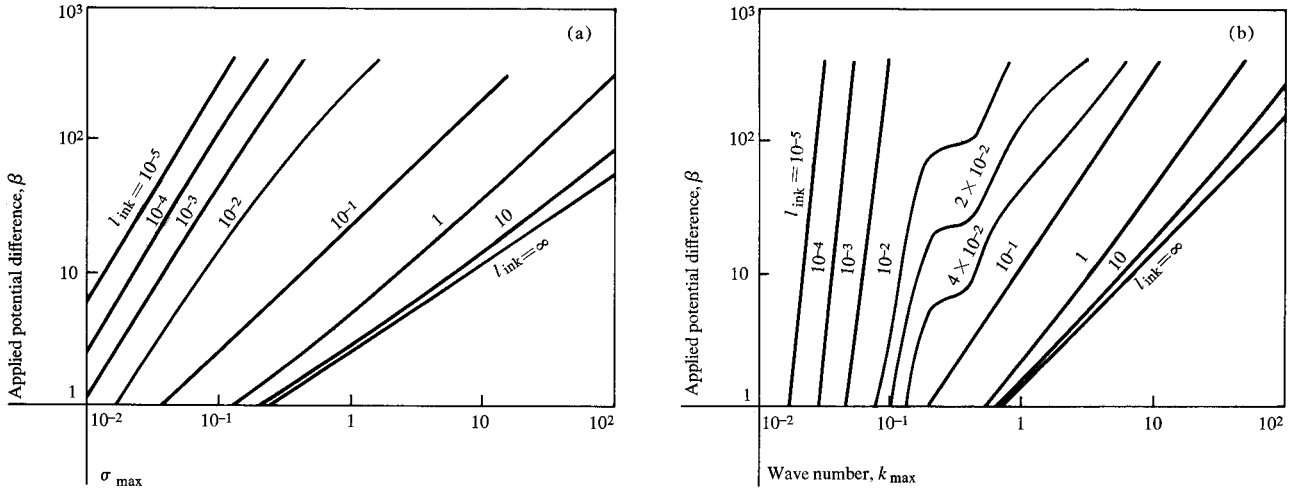

Figure 5 Unstable disturbances for inviscid oil as function of applied potential difference. (a) Maximum growth rate. (b) Wave number corresponding to the maximum growth rate.

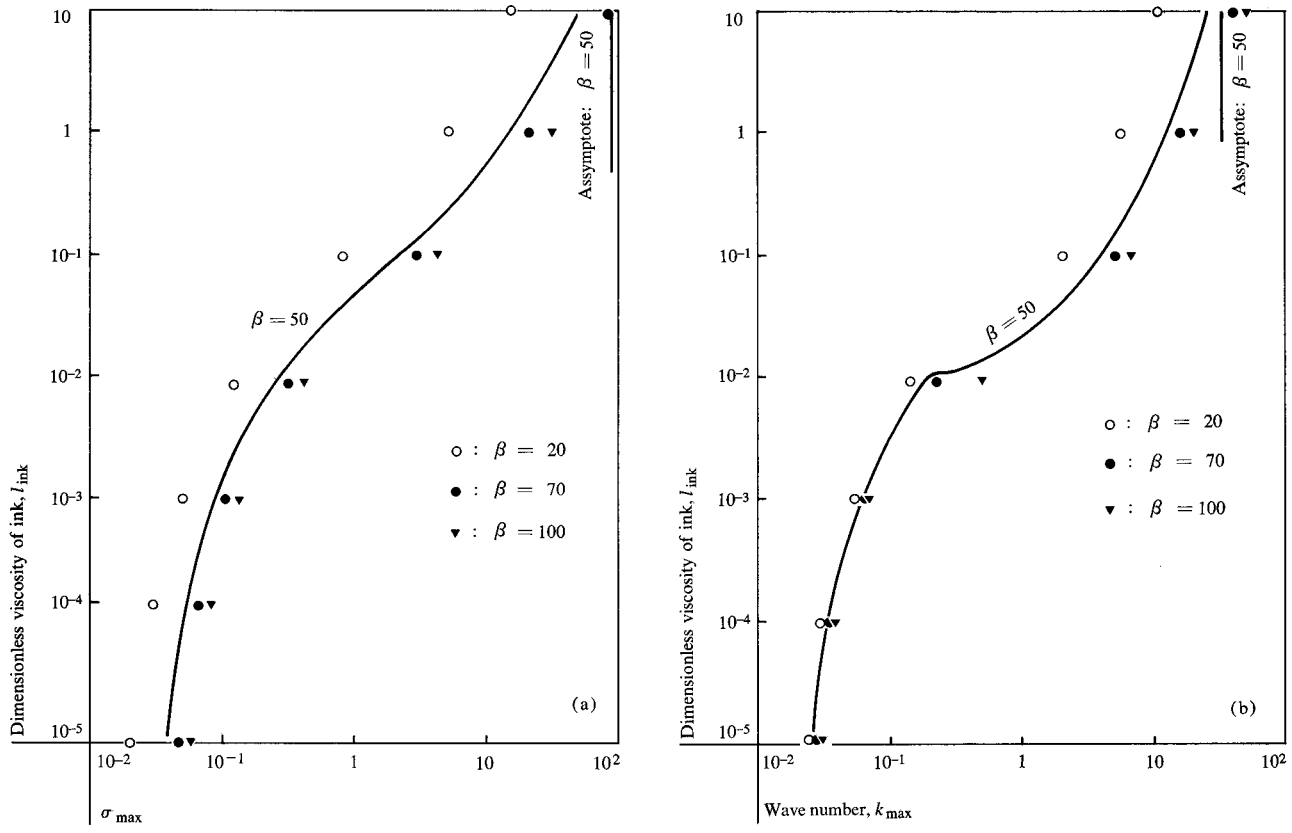

Figure 6 Unstable disturbances for inviscid oil as function of the dimensionless viscosity of ink. (a) Maximum growth rate. (b) Wave number corresponding to the maximum growth rate.

$$\phi''(x_3) - k^2\phi(x_3) = 0,$$

$$\rho a_0 n v_3(x_3) = -\phi'(x_3),$$

$$\bar{\phi}''(x_3) - k^2\bar{\phi}(x_3) = 0,$$

$$\bar{v}_3''(x_3) - \bar{\lambda}^2\bar{v}_3(x_3) = (a_0/\bar{\mu})\bar{\phi}'(x_3),$$

$$\bar{\lambda}^2 = k^2 + (\bar{\rho}a_0^2 n/\bar{\mu}). \tag{41}$$

The boundary conditions are (27a, b), (28b), (29b), and (30a, b). The solution of the homogeneous system

leads to the characteristic equation given by a 6×6 determinant. Numerically, for a given k, we have to find the zero of the determinant.

Computations have been carried out for the parameter $l_{ink} = 10^{-5}$, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 1, and 10. (When $l_{ink} > 10^2$, the results practically coincide with those of the inviscid oil and ink.) Quite drastic reduction in the growth rate due to the viscosity of the ink is observed for large wave numbers. This is obvious in Figs. 4, and 7(a), where the growth rate is plotted for $\beta = 40$, 60, and 50, respectively. Consequently, the wave number k_{max} corresponding to σ_{max} is shifted towards decreasing k. However, the viscosity of the ink has little effect upon the growth rate for small wave numbers. The maximum growth rate σ_{max} is given in Figs. 5(a) and 6(a) as functions of β and l_{ink}, respectively. The wave number k_{max} that corresponds to σ_{max} is plotted in Figs. 5(b) and 6(b). There seems to be some sharp increase in k_{max} when $l_{ink} \approx 10^{-2}$. This is realizable from Figs. 4 and 7(a). There is a fairly flat region of each growth rate curve when $l_{ink} < 10^{-2}$. As l_{ink} increases and passes 10^{-2}, this flat region moves upward to reach and to pass σ_{max}. Thus, numerically, there may be difficulty to determine precisely the location of k_{max} when $l_{ink} \approx 10^{-2}$.

• *Viscous ink and oil*

Taking into consideration the viscosities of both the ink and the oil, the homogeneous problem involves the full equations (27) through (32). The characteristic equation is given in the form of an 8×8 determinant. It is found, analytically, that the conclusions 1, 2, 3, and 4 from the results for the inviscid oil and ink are equally valid. Computations have been carried out for the cases $\beta = 50$, $l_{ink} = 10^{-3}$ and 10^{-4} and for the values of $l_{oil} = 10^3$, 10^2, 10, 1, and 10^{-1}. The results are plotted in Fig. 7. A contrasting phenomenon is observed apparently in Fig. 7(a). The viscosity of the ink reduces the growth rate quite drastically for large wave numbers, and it has little effect on waves of small wave numbers. On the contrary, the viscosity of the oil has little effect on waves of large wave numbers, but it tends to stabilize waves of small wave numbers.

Summary

When an electric field is present across the oil in the liquid ink development process of electrophotography, some disturbances in the oil-ink interface are found to grow in time. If enough time is allowed in the developing stage, the ink may eventually break through the thin oil film. When the disturbance is analyzed into normal modes, such growing, hence unstable, modes of disturbances are found to be confined in a finite range of wave numbers. There exists a wave number at which the growth rate is maximum. For a given mode, there exists a threshold potential difference across the oil below which the wave is stable.

The finite viscosity of the ink reduces the growth rate quite drastically at large wave numbers, and it has little effect on waves of small wave numbers. On the contrary, the effect of oil viscosity has little effect on waves of large wave numbers, but it tends to stabilize waves of small wave numbers.

Formally in the solution, the pressure at the ink-elastomer interface does not appear explicitly in this linear analysis. The effect of the pressure is, however, felt indirectly through its influence on the thickness of the oil film. Such an influence may be established by other means independently of the present analysis.

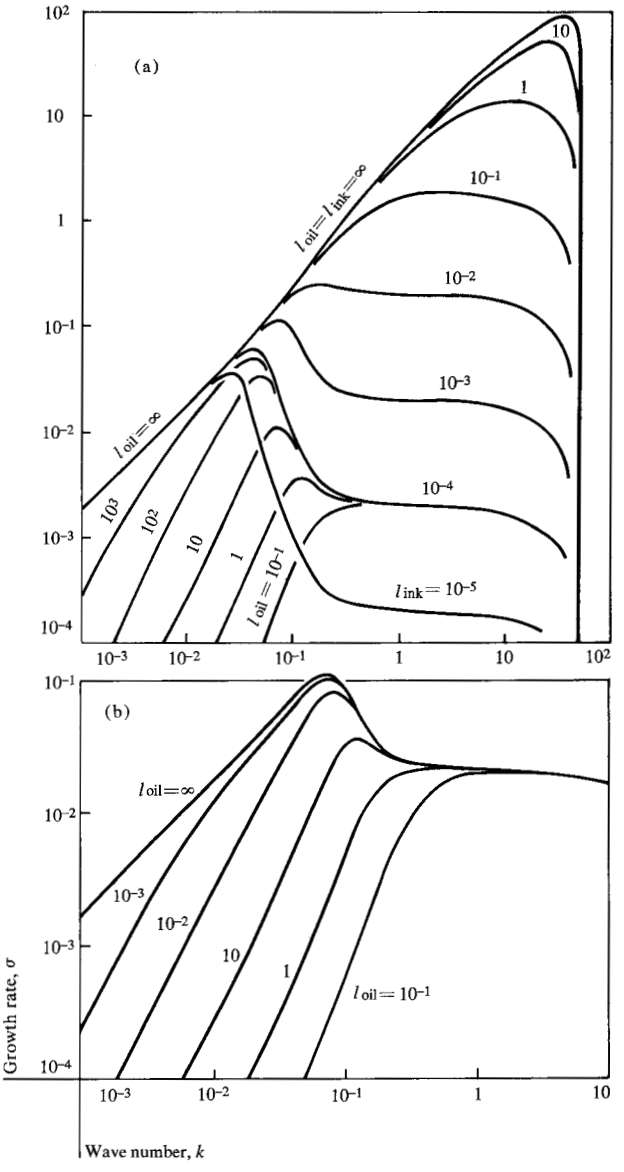

Figure 7 Dimensionless growth rates for $\beta = 50$. (a) Details for viscous oil when $l_{ink} = 10^{-4}$. (b) Details for viscous oil when $l_{ink} = 10^{-3}$.

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